Assessment Of The Spatial Variability Of Soil Chemical Properties Along A Transect Using Multifractal Analysis

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Abstract

The spatial variability of soil properties can be assessed through concepts of scale invariance, fractals and multifractals. The aim of this study was to characterize the scaling patterns and structural heterogeneity properties of general soil chemical properties along a short (i.e. 52 m large) transect. Field measurements were carried out at the experimental farm of CIAM located in Mabegondo, A Coruña, Spain. The studied transect was marked following land slope, and 66 soil samples were collected at the 0-20 cm depth every 0.8 m. The soil properties analyzed were: pH (H_2O), organic carbon content, exchangeable Ca, Mg and K, exchangeable acidity (H + Al), exchangeable bases (SB), cation exchange capacity (CEC), percent base saturation (V) and extractable P. The soil properties studied showed various degrees of multifractality. The spatial distribution of pH was characterized by quasi-monofractal behaviour; CEC, (H+Al) and OM, presented a relatively low degree of multifractality, and the other soil properties studied showed stronger degrees of multifractality, being the highest one for Olsen extractable P. In general, the scaling features of the properties studied implied a multifractal nature, where the low and high density regions scaled differently.

Key words: spatial variability, soil chemistry, nutrients, multifractal analysis

1. INTRODUCTION

Spatial variability in earth sciences has been widely analyzed for different types of sampling schemes. Although samples have been more frequently collected in grids over areal surfaces (Caridad Cancela et al., 2005; Morales et al., 2010, 2014), several studies also performed sampling along transects (Caniego et al., 2005; Vidal-Vázquez et al., 2013), mainly to appraise localized variability.

Variability of soil properties stems from interactions between the five factors of soil formation (climate, parent material, topographic relief, organisms and time) and a number of physical, mineralogical, chemical, biological and environmental processes that most often do not run independently. Moreover, soil forming factors and soil processes can operate at various scales, so that the spatial variability of soil properties may exhibit nested effects. Consequently, either soil properties that are constant over time (such as particle size distributions, particle density, and to some extent also cation exchange capacity or organic carbon content) or those highly dynamic (water content, bulk density, porosity, nutrient status, fungal and bacterial biomass, etc.) can display a rich spatial (Vieira et al., 1983, 2010; Vieira and Paz González, 2003; Dafonte et al., 2010; Caridad-Cancela et al., 2005; Montanari et al., 2011) or spatial and temporal (Morales et al., 2010, 2014, Vidal Vázquez et al., 2012) variability.

Scaling analysis, such as fractal and multifractal analysis, also has been also used to adequately characterize the spatial variability of soil properties (Caniego et al., 2005; Zeleke and Si, 2006; Vidal Vázquez et al., 2013) and to describe the combination of irregularity and structure of such properties for a large range of scales (Tarquis et al., 2008; Vidal Vázquez et al., 2008a, b; Siqueira et al., 2013; Paz-Ferreiro et al., 2008, 2010, 2014). While in monofractal scaling one single exponent is sufficient to capture the scaling behavior of the studied data set, multifractal scaling involves entire functions, which represents a hierarchy of exponents related to different levels of intensity or irregularities of the data series (Everstz and Mandelbrot, 1992; Falconer, 1997). Thus, multifractals are intrinsically more complex and inhomogeneous than monofractals. Both, fractal and multifractal scaling assume a hierarchical distribution of mass in space, so that the whole results from the union of similar subsets (Caniego et al., 2005; Tarquis et al., 2008).

The multifractal approach has been proved to be useful for characterizing the inner structure of soil general properties measured along transects ant to compare them (Caniego et al., 2005; Zeleke and Si., 2006; Vidal Vázquez et al., 2013). Moreover, multifractal analysis has been demonstrated to provide more information about the inner structure and the spatial variability of a data set than semivariogram analysis (e.g. Kravchenko, 2008). This is because multifractal characteristics allow the assessment of anomalous behaviour of singular physical processes. The objective of this research was to quantify the scaling patterns and structural heterogeneity of various soil chemical properties measured in a small transect using multifractal spectra and indices.

2. MATERIAL AND METHODS

2.1. Experimental site and sampling

Soil samples were collected at an experimental field of the "Centro de Investigaciones Agrarias de Mabegondo" (CIAM), A Coruña province, Spain, (Latitude 43° 14' 47" N, Longitude 8° 16' 23" W). The site was on a gentle slope and the soil was a loamy textured Inceptisol (SSA, 2010). Sixty six soil samples were collected at 0 - 20 cm depth every 0.8 m along a transect, which was marked following the land slope. Thus the total length of the sampled transect was 52 m.

2.2. Analysis of general soil properties

Soil samples were air-dried, crushed and sieved through a 2 mm mesh. Soil pH, organic carbon content, exchangeable bases ($Ca^{+2} + Mg^{+2} + K^+$) exchangeable acidity (H^+ plus Al^{+3}), and extractable phosphorus were analyzed. In addition, cation exchange capacity (CEC), sum of exchangeable bases (SB = $Ca^{+2} + Mg^{+2} + K^+$) and percent base saturation (V%) were computed from exchangeable cations. Organic carbon content was determined following the Walkley-Black method (van Raij et al. 2001; da Silva Días et al., 2013). Phosphorus was determined colorimetricaaly after extraction by the resin method (van Raij et al. 2001; Paz-Ferreiro et al., 2012). Exchangeable bases ($Ca^{+2} + Mg^{+2} + K^+$) were determined after extraction with a cation exchange resin, whereas extractable acidity ($H^+ + Al^{+3}$) was determined after Al extraction with KCl, as described in van Raij et al. (2001).

2.3. Multifractal analysis

Multifractal analysis was implemented following the moment method, which is next summarized. First, a mesh with size δ was employed to be superimposed over the whole support. In other words, the length of the transect was divided into smaller and smaller segments based on dyadic downscaling. This was implemented by successive partitions of the support in k stages (k=1, 2, 3…) that generate at each scale, δ a number of segments, $N(\delta) = 2^k$ of characteristic size length, $\delta = L \times 2^{-k}$, covering the whole extent of the support, L, in this case a transect (e.g. Evertsz and Mandelbrot, 1992; Caniego et al., 2005; Vidal Vázquez et al., 2013).

Then, the experimental data for each variable studied were converted into the distribution of mass along the geometric support. Therefore, the probability mass function, $pi(\delta)$, for each segment was estimated as a proportion according to:

$$p_i(\delta) = \frac{N_i(\delta)}{N_t} \qquad (1)$$

where $N_i(\delta)$ is the value of the measure in a given segment, i^{th}, and N_t is the sum of the measure in the whole transect.

Multifractal analysis involves several scaling functions: mass exponent, τ_q, singularity spectrum, $f(\alpha)$, local scaling index, α_q, and generalized or Rényi dimension, D_q. In practice, using the box counting method, the so-called partition function scales with the segment size as follows:

$$\chi(q, \delta) = \sum_{i=1}^{n(\delta)} p_i^q(\delta) \cdot \qquad (2)$$

Where $n(\delta)$ is the number of segments with size δ and statistical moments q are defined for $-\infty < q < \infty$.

A log-log plot of the quantity $\chi(q, \delta)$ versus δ for different values of q yields: $\chi(q, \delta) \propto \delta^{-\tau(q)}$, where τ_q is the mass scaling function of order q. Note that the method of moments is justified if the plots of $\chi(q, \delta)$ versus δ are straight lines (Halsey et al., 1986).

The mass exponent function τ_q was estimated from the partition function as:

$$\tau(q) = \lim_{\delta \to 0} \frac{\log \chi(q, \delta)}{\log(1/\delta)} \qquad (3)$$

The function τ_q controls how the moment of measure μ_i scales with q. In general, multifractal measures yield a nonlinear function of τ_q, whereas a monofractal corresponds to linear τ_q.

For each box or segment, the probability distribution is: $p_i(\delta) = \delta^{\alpha i}$, where α_i is the singularity or Hölder exponent characterizing density in the ith box (Halsey et al., 1986). The Hölder exponent, given by $\alpha^i = \log \mu_i(\delta)/\log \delta$, may be interpreted as a crowding index for the degree of concentration of the measure, μ. It is, in fact, the logarithmic density of the i-th box of the partition of characteristic size δ.

For multifractal distributed measures, the number $N_\delta(\alpha)$ of cells of size δ having a singularity or Hölder exponent equal to α increases for decreasing δ and obeys a power law: $N(\alpha) \propto \delta^{-f(\alpha)}$, where the exponent $f(\alpha)$ is a continuous function of α. The graph of $f(\alpha)$ versus α, called the multifractal spectrum, typically has a parabolic concave downward shape, with the range of α-values increasing with the increase in the heterogeneity of the measure. The minimum scaling exponent $f(\alpha_q)$ corresponds to the most concentrated region of the measure, and the maximum exponent $f(\alpha_{q+})$ corresponds to the rarefied regions of the measure. The connection between the scaling exponents τ_q and $f(\alpha)$ can be made through a Legendre transformation. In this work, however, the functions t_q and $f(\alpha)$ were obtained following Chhabra and Jensen (1989) with the equations:

$$\alpha(q) \propto \frac{\sum_{i=1}^{N(\delta)} \mu_i(q, \delta) \log[\mu_i(\delta)]}{\log(\delta)} \qquad (4a)$$

$$f(\alpha(q)) \propto \frac{\sum_{i=1}^{N(\delta)} \mu_i(q, \delta) \log[\mu_i(q, \delta)]}{\log(\delta)} \qquad (4b)$$

The scaling function, τ_q is also related to the generalized fractal dimension (Hentschel and Procaccia, 1983), which can be defined by Equations 5a. In fact, the concept of generalized dimension, D_q, corresponds to the scaling exponent for the qth moment of the measure. Moreover, the generalized dimensions can be also defined by Equation 5b. Note, however, that using Equations 5a or 5b D_1 becomes indeterminate because the value of the denominator is zero. Therefore, for the particular case that $q = 1$, Equation 5c is used.

$$D_q = \tau(q)/(q-1) \qquad (5a)$$

$$D_q = \lim_{\delta \to 0} \frac{1}{q-1} \frac{\log[\chi(q, \delta)]}{\log \delta} \qquad (5b)$$

$$D_1 = \lim_{\delta \to 0} \frac{\sum_{i=1}^{n(\delta)} \chi_i(1, \delta) \log[\chi_i(1, \delta)]}{\log \delta} \qquad (5c)$$

For a monofractal, D_q is a constant function of q, so no additional information is obtained by examining higher moments. However, for multifractal measures, the relationship between D_q and q is not constant. In this case, the most frequently used gener-

alized dimensions are D_0 for $q = 0$, D_1 for $q = 1$ and D_2 for $q = 2$, which are referred to as capacity, information (Shannon entropy) and correlation dimension, respectively.

The capacity or box-counting dimension, D_0, is the scaling exponent of the number of non-empty segments. Thus, it is independent of the quantity of mass in each box, but takes into account the fact that the segments are occupied or not. The information dimension, D_1, gives the probability of occupation of the ith segment of size δ, without taking into account the way in which the measure is distributed within each of these segments. Thus, D_1 provides a physical characterization, indicating how heterogeneity changes across a certain range of scales, and it is also related to the Shannon entropy. The correlation dimension, D_2, describes the uniformity of the measure values among intervals. The generalized dimension, D_q, is widely used for the comprehensive study of multifractals. Differences between D_q allow comparison of the complexity between the studied data sets. The higher the homogene-

ity of a structure, the closer the D_q values; thus in a monofractal D_q is constant.

3. RESULTS AND DISCUSSION

3.1. Statistical analysis

A summary statistical analysis of the studied soil properties is listed in Table 1. Mean soil pH was 4.9 and overall pH values along the sampled transect varied from 4.6 to 5.4 along, which means very strongly acid to strongly acidic soil reaction. Mean organic matter content was 41.7 mg kg[-1], and the range of OM values varied from 28.0 to 62.0 mg kg[-1]. Cation exchange capacity ranged from 9.37 to 15.06 Cmol+kg[-1], with a mean value of 12.23 Cmol+kg[-1], and the sum of exchangeable bases was lower than exchangeable acidity, as expected from the low pH values. Subsequently percent base saturation was lower than 50%. Olsen extractable phosphorus content ranged from 6.0 to 39.0 mg kg[-1].

Coefficients of variation were highest for Olsen extractable P and lowest for pH, and

|	mean	std	c.v.	variance	maximum	minimum
pH	4.9	0.16	3.3	0.03	5.4	4.6
OM (g.kg-1)	41.7	7.76	18.6	60.23	62.0	28.0
CEC (Cmol+kg-1)	12.23	11.51	9.4	132.52	15.06	9.37
SB (Cmol+kg-1)	3.22	9.08	28.2	82.36	6.11	1.94
Ca (Cmol+kg-1)	2.54	7.72	30.4	59.59	0.50	1.50
Mg (Vmol+kg-1)	0.32	1.39	43.3	1.93	0.69	0.13
K (Cmol+kg-1)	0.36	0.72	20.1	0.51	0.56	0.23
H+Al (Cmol+kg-1)	9.01	11.84	13.1	140.07	12.10	5.80
V (%)	26.3	6.94	26.4	48.14	44.0	14.0
P (mgkg-1)	12.5	6.26	50.3	39.17	39.0	6.0

Table 1. Summary statistics of the studied soil properties measured along a transect (std = standard deviation, c.v.= coefficient of variation).

ranked as follows: P> Mg > Ca > SB > V > K > OM > H + Al > CEC > pH. Therefore, Olsen-P showed a CV higher than 50%; moreover exchangeable bases and the attributes of the exchange complex SB an V exhibited a relatively high statistical variability with CV's higher than 20%. Previous studies also have reported high and low values of spatial variability for exchangeable bases (Vidal Vázquez et al., 2013) and pH (Caniego et al., 2005; Vidal Vázquez et al., 2013), respectively.

3.2. Multifractal analysis.

The distribution of a measure is considered fractal (mono- or multifractal) when the partition function for successive moments can be fitted by power law functions (Evertsz and Mandelbrot, 1992; Zeleke and Si, 2006). Therefore, plots of the normalized measure χ (q,δ) versus measurement scale, δ, were examined, for all the statistical moments of interest, to find out whether the studied properties obeyed or not power law scaling. For moment orders in the range between $q = +5$ and $q = -5$, the logarithm of $\chi(q,\delta)$ versus the logarithm of δ fitted a linear model when the partition function was constructed for successive box sizes in steps of 2^k, k=0 to k=7. Consequently, partition functions have been estimated in the range of linear behaviour, involving box sizes limited to 0< k <7.

pH
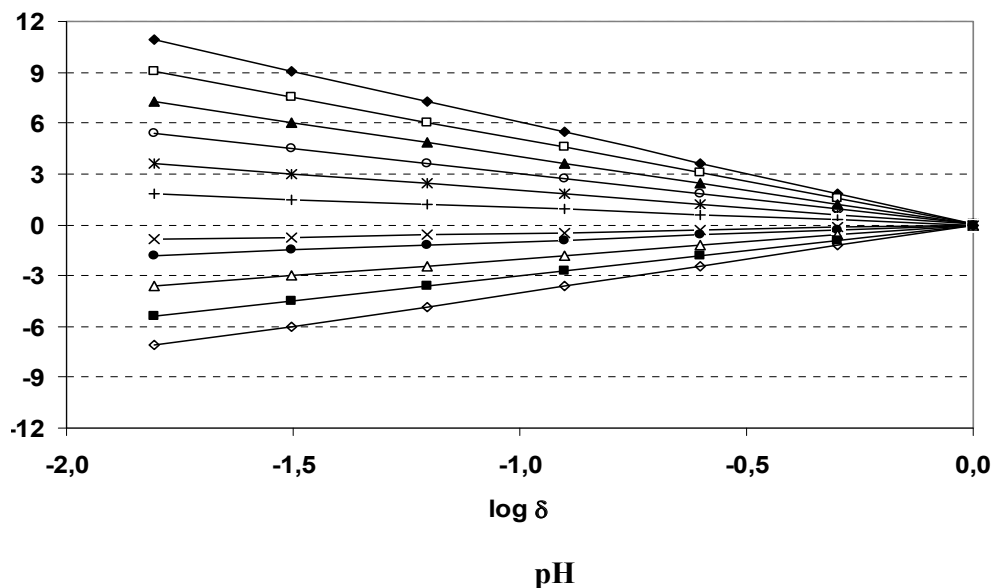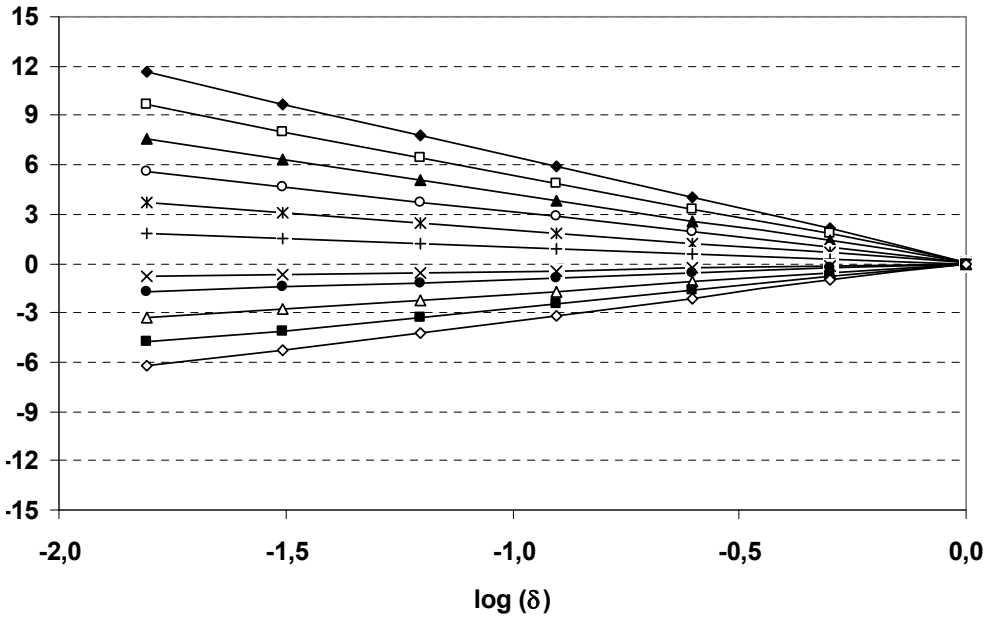
Olsen P

Figure 1. Examples of log–log plots of the partition function, χ (q,δ),versus measurement scale, δ, for pH and Olsen extractable P.

Examples of partition functions are shown in Figure 1. The log-log plots of χ(q,δ)-δ always fitted a linear model with high and coefficients of determination (P<0.05), and there were some differences in the goodness of fit to a power law between the variables studied.

The singularity spectrum was estimated by Equations 4a and 4b. The range of statistical moments used to compute f(α)-α plots was = -5.0 < Δq < 5.0 for all the data sets studied. Singularity spectra were concave down parabolic curves with more or less asymmetry, and for all the studied cases the left branch was longer than the right branch. Selected examples of singularity spectra are shown in Figure 2. Shape, amplitude and symmetry of the f(α)-α plots in Figure 2 show wide differences, which will be next addressed for testing the hypothesis of singular behavior of the measured soil properties.
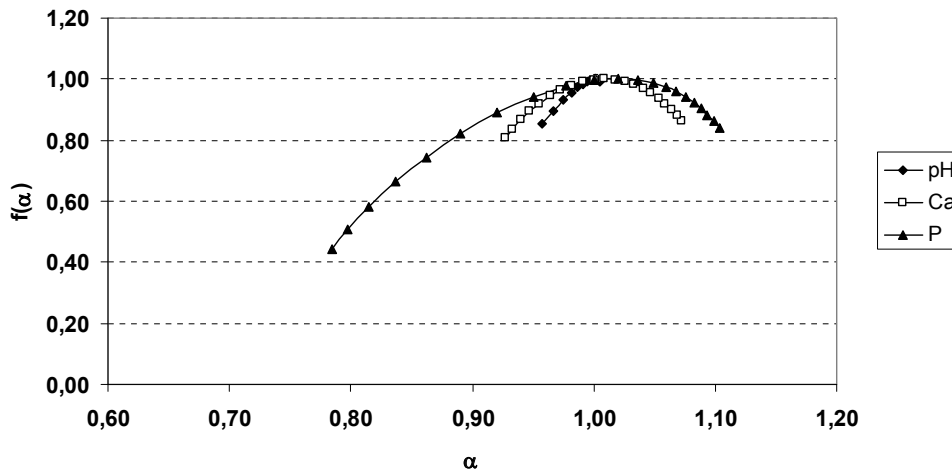
Figure 2. Singularity spectra for pH, exchangeable Ca and extractable P.

Several multifractal parameters obtained from the singularity spectra are listed in Table 2. The Hölder exponent of order zero, α_0, varied between 1.001 and 1.020 and determination coefficients in estimating α_0 were 1.000. Moreover, when calculating α_{-5} and α_{+5}, determination coefficients were 0.996 and 0.986, respectively. Goodness of fit statistics of the singularity spectra showed the scaling properties of the studied soil variables can be fitted reasonably well with fractal or multifractal models.

Width or amplitude of the singularity spectrum, $(\alpha_{q-} - \alpha_{q+})$, in this case, $(\alpha_{-5} - \alpha_5)$, is an indicator of heterogeneity, because it provides information on the diversity of the scaling exponents of a measure. So, the wider the f(α) spectrum is, the higher is the heterogeneity in the scaling indices. Differences in the width of the measure assessed as $(\alpha_{-5} - \alpha_5)$ ranged from 0.047 for pH to 0.0319 for Olsen extractable P.

The wider the f(α)-α spectrum is, the higher is the heterogeneity in the local scaling indices of the study variable and vice versa. Moreover, in a homogeneous fractal system the f(α) spectra would be reduced to a single point. Therefore, Figure 2 and Table 2 show that the studied variables have very different degrees of multifractality.

Variable	q_-	q_+	$(\alpha_{-5} - \alpha_5)$	α_0	α_{-5}	α_5
pH	-5	+ 5	0.047	1.001 ± 0.000	1.005 ± 0.000	0.958 ± 0.021
OM	-5	+ 5	0.094	1.005 ± 0.001	1.043 ± 0.006	0.949 ± 0.019
CEC	.5	+ 5	0.055	1.002 ± 0.001	1.013 ± 0.004	0.958 ± 0.026
SB	-5	+ 5	0.238	1.011 ± 0.003	1.068 ± 0.022	0.830 ± 0.070
Ca	-5	+ 5	0.146	1.009 ± 0.002	1.073 ± 0.016	0.927 ± 0.012
Mg	-5	+ 5	0.222	1.017 ± 0.008	1.105 ± 0.050	0.883 ± 0.043
K	-5	+ 5	0.150	1.006 ± 0.001	1.045 ± 0.010	0.895 ± 0.051
H+Al	-5	+ 5	0.078	1.003 ± 0.001	1.029 ± 0.012	0.950 ± 0.032
V	-5	+ 5	0.179	1.009 ± 0.002	1.070 ± 0.020	0.891 ± 0.040
P	-5	+ 5	0.319	1.020 ± 0.003	1.103 ± 0.025	0.784 ± 0.019

Table 2. Parameters and indices obtained from the singularity spectrum.

In addition, shape and asymmetry of the $f(\alpha)$-α spectrum can be employed to assess the heterogeneity of the measure. Note also that the presence of extremely high and extremely low data values and dominance of either low or high data are related to the left ($q \gg 1$) and right ($q \ll 1$) parts of the $f(\alpha)$ spectrum, respectively. For the studied soil properties measured along profiles, the left branch of the $f(\alpha)$-α spectrum was wider than the left branch. Asymmetry toward the left indicates dominance of the lowest singularity exponents, α. In general, the left side was also longer than the right side, revealing that the geometrical size of points with the smallest exponents, α was smaller. The opposite was true for the narrower and shorter right side of the singularity spectrum. This suggest dominance of lowest values along de measured profiles and that these were quite similar to each other, as compared to the highest values that were less frequent and showed more differences between one another along the profile.

Selected examples of generalized dimension spectra are shown in Figure 3. They follow a typical sigma shaped curve that crosses through 1.0 at $q=0$. The amplitude of the D_q curvature, however, varied widely and it was very narrow for pH, intermediate for exchangeable Ca and rather wide for Olsen-P. Standard errors of D_q also increased with increasing |q|, and their maximum values for $q=10$, $q=1$, $q=2$, $q=5$ and $q=-5$ were as high as ± 0.000, ± 0.005 ± 0.005, ± 0.021 and ± 0.019, respectively. Determination coefficients were highest for $q = 0$, ($R^2=1.000$) and decreased with increased |q|; for $q = 1$, $q =2$, $q = 5$, and $q = -5$, the value of R^2 was higher than 1,000, 0.999, and 0.997 and 0.998, respectively (data not shown).

The curvature of the generalized dimension, Dq, was always much higher for positive than for negative values of q. This branch of the Rényi spectra corresponds to the smallest concentrations of the measure. This is in accordance with the fact that the singularity spectra, f(α)-α, had a wider left branch.
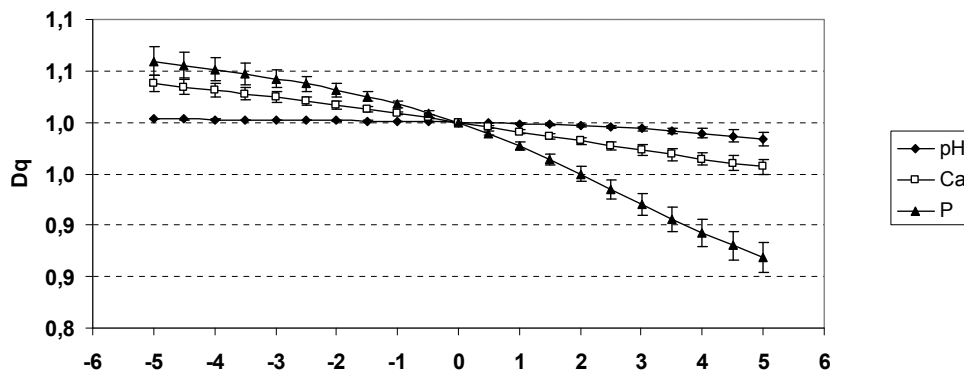

Figure 3. Generalized dimension.

The generalized dimension function also provides information to assess whether the scaling properties observed correspond to monofractal o multifractal behaviour. The spatial distribution of the soil properties studied would have a tendency to follow multifractal type of scaling if $D_0 > D_1 > D_2$. Table 3 shows that for pH, $D_0 \approx D_1 \approx D_2$, as the difference between D_0 and D_2 were lower than 0.05. Moreover, for CEC, (H+Al) and OM, values of ($D_0 - D_2$) were lower than 0.01. However the remaining variables can be regarded as multifractal measures. This result evidences various degrees of multifractality of the studied soil properties.

Variable	(D_{-5} - D_5)	D_{-5}	D_5	D_0	D_1	D_2
pH	0,019	1.003 ± 0.000	0.984 ± 0.004	1.000 ± 0.000	0.999 ± 0.000	0.997 ± 0.001
OM	0.047	1.021 ± 0.002	0.974 ± 0.005	1.000 ± 0.000	0.995 ± 0.001	0.990 ± 0.002
CEC	0.024	1.006 ± 0.001	0.982 ± 0.006	1.000 ± 0.000	0.998 ± 0.001	0.995 ± 0.001
SB	0.125	1.037 ± 0.007	0.912 ± 0.021	1.000 ± 0.000	0.988 ± 0.002	0.973 ± 0.004
Ca	0.081	1.038 ± 0.005	0.957 ± 0.005	1.000 ± 0.000	0.991 ± 0.001	0.982 ± 0.002
Mg	0.135	1.059 ± 0.019	0.924 ± 0.019	1.000 ± 0.000	0.982 ± 0.005	0.964 ± 0.010
K	0.072	1.023 ± 0.003	0.951 ± 0.013	1.000 ± 0.000	0.994 ± 0.001	0.986 ± 0.002
H+Al	0.035	1.013 ± 0.003	0.978 ± 0.008	1.000 ± 0.000	0.997 ± 0.001	0.994 ± 0.002
V	0.092	1.035 ± 0.006	0.943 ± 0.011	1.000 ± 0.000	0.991 ± 0.001	0.980 ± 0.003
P	0.191	1.059 ± 0.009	0.868 ± 0.009	1.000 ± 0.000	0.977 ± 0.002	0.950 ± 0.005

Table 3. Parameters and indices obtained from the generalized dimension.

Following previous criteria (Vidal Vázquez et al. 2013) the spatial distribution of pH is characterized by quasi-monofractal behaviour, CEC, (H+Al) and OM, present a relatively low degree of multifractality, and the other soil properties studied (exchangeable K, Mg and Ca, SB, V and Olsen-P) show a stronger degrees of multifractality.

The difference ($D_{-5} - D_5$) also has been frequently used as an index of multifractality. The value of this parameter ranged from 0.019 (pH) to 0.191 (Olsen-P). Therefore, again analysis of D_q spectra showed that the spatial distribution of the studied soil attributes exhibited various degrees of scaling heterogeneity or multifractality.

The sampled transects was representative of the main pedological conditions along a slope at the studied site. The spatial variability of this transect, sampled reflects the local effects of the main soil forming factors and processes. Therefore, the scaling properties of the soil attributes embody a realistic situation, consistent with observations at the small plot level. At this level, the multifractal approach gives a good description of the spatial variability along transects.

CONCLUSIONS

We investigated the multifractal properties of several soil chemical properties, including pH, OM, attributes of the soil exchange complex and Olsen extractable P sampled along a transect transects on a gentle slope. All the singularity spectra studied were characterized by a wider left branch, whereas the generalized dimension spectra had a wider right branch, which correspond to the smallest concentrations of the measure.

Singularity spectrum, f(α)-α, and generalized dimension, Dq, curves showed that the spatial distributions of these soil properties exhibited various degrees of multifractality. Thus, the spatial distribution of pH was characterized by quasi-monofractal behaviour, CEC, (H+Al) and OM, showed a relatively low degree of multifractality, and the other soil properties studied (exchangeable K, Mg and Ca, SB, V and Olsen-P) exhibited a stronger degrees of multifractality.

REFERENCES

CANIEGO, F.J., ESPEJO, R., MARTÍN, M. A., SAN JOSÉ, F. (2005). Multifractal scaling of soil spatial variability. Ecological Modelling, 182:291-302.

CARIDAD-CANCELA, R., VIDAL VÁZQUEZ, E., VIEIRA, S. R., ABREU, C. A., PAZ GONZÁLEZ, A. (2005). Assessing the spatial uncertainty of mapping trace elements in cultivated fields. Communications in Soil Science and Plant Analysis, 36(1,3): 253-274.

CHHABRA A.B., JENSEN, R. V. (1989). Direct determination of the f(α) singularity spectrum. Physical Revew Letters, 62: 1327-1330.

DAFONTE, DAFONTE, J., GUITIÁN OJEA, M., PAZ-FERREIRO, J., MACHADO SIQUEIRA, G., VIDAL VÁZQUEZ, E. (2010). Mapping of soil micronutrients in an European Atlantic agricultural landscape using ordinary kriging and indicator approach. Bragantia, 69: 175-186.

EVERSTZ, C.J.G., MANDELBROT, B. B. (1992). Multifractal measures. In Peitgen, H., H. Jürgens, and D. Saupe.

Chaos and Fractals, Springer, Berlin, 921-953.

FALCONER, K. (1997). Fractal Geometry. Mathematical Foundations and Applications. John Wiley and Sons, Chichester.

HALSEY, T.C., JENSEN, M. H., KADANOFF, L. P., PROCACCIA, I., SHRAIMAN, B. I. (1986). Fractal measures and their singularities: The characterization of strange sets. Physical Review A, 33: 1141-1151.

HENTSCHEL, H.G.E., PROCACCIA, I. (1983). The infinite number of generalized dimensions of fractals and strange attractors. Physica D, 8: 435-444.

KRAVCHENKO, A.N. (2008). Stochastic simulations of spatial variability based on multifractal characteristics. Vadose Zone Journal, 7(2):521-524.

MONTANARI, R., LIMA, R.C., BONINI, A.S., MARUES, L.S., MINGUINI, R., CARVALHO, M.P., PAZ-FERREIRO, J., COSTA, N.R. (2011).Variabilidade dos atributos de um Latossolo vermelho sob plantio direto no cerrado brasileiro e produtividade da soja. [Variability of the features of a red latosol on direct yield in a Brazilian cerrado and soybean productivity]. Cadernos do Laboratorio Xeoloxico de Laxe, 36: 219-237.

MORALES, L.A., PAZ-FERREIRO, J., VIEIRA, S.R., VIDAL VÁZQUEZ, E. (2010). Spatial and temporal variability of EH and PH over a rice field as related to lime addition. Bragantia, 69: 67-76.

MORALES, L.A., VIDAL VÁZQUEZ, E., PAZ-FERREIRO, J. (2014). Spatial distribution and temporal variability of ammonium-nitrogen, phosphorus, and potassium in a rice field in Corrientes, Argentina. Scientific World Journal. DOI.org/10.1155/2014/135906

PAZ-FERREIRO, J., BERTOL, I., VIDAL VÁZQUEZ, E. (2008). Quantification of tillage, plant cover, and cumulative rainfall effects on soil surface microrelief by statistical, geostatistical and fractal indices. Nonlinear Processes in Geophysics, 15 (4): 575-590.

PAZ-FERREIRO, J., MIRANDA, J.G.V. and VIDAL VÁZQUEZ, E. (2010). Multifractal analysis of soil porosity based on mercury injection and nitrogen adsorption. Vadose Zone Journal, 9 (2): 325-335.

PAZ-FERREIRO, J, VIDAL VÁZQUEZ, E. and de ABREU, C.A. (2012). Phosphorus Determination after Mehlich 3 Extraction and Anion Exchange Resin in an Agricultural Soil of Northwestern Spain. Communications in Soil Science and Plant Analysis, 43: 102-111.

PAZ-FERREIRO, J. and VIDAL VÁZQUEZ, E.V. (2014). Pore size distribution patterns in tropical soils obtained by mercury intrusion porosimetry: The multifractal approach. Vadose Zone Journal, 13 (6). DOI: 10.2136/vzj2014.01.0003

SILVA DÍAS (da), R., de ABREU, C.A., de ABREU, M.F., PAZ-FERREIRO, J., MATSURA, E.E., and PAZ GONZÁLEZ, A. (2013). Comparison of Methods to Quantify Organic Carbon in Soil Samples from São Paulo State, Brazil. Communications in Soil Science and Plant Analysis, 44: 429-439.

SIQUEIRA, G. M., SILVA, E.F.F., MONTENEGRO, A.A.A.,VIDAL VÁZQUEZ, E. and PAZ-FERREIRO, J. (2013). Multifractal analysis of vertical profiles of soil penetration resistance at the field scale. Nonlinear Processes in Geophysics, 20 (4): 529-541.

SOIL SURVEY STAFF. (2010). Keys to Soil Taxonomy. 11th Edition. Natural Resources Conservation Service. Washington. DC. 338 pp.

TARQUIS, A.M., BIRD, N.R.A. WHITMORE A.P., CARTAGENA, M.C., and PACHEPSKY. Y. 2008. Multiscale Entropy-based Analysis of Soil Transect Data. Vadose Zone Journal, 7:563-569.

VIDAL VÁZQUEZ, E., PAZ-FERREIRO J., MIRANDA, J.G.V. and PAZ GONZÁLEZ, A. (2008a). Multifractal analysis of pore size distributions as affected by simulated rainfall. Vadose Zone Journal, 7 (2): 500-511.

VIDAL VÁZQUEZ, E., GARCÍA MORENO, R., MIRANDA, J.G.V., DÍAZ, M.C., SAÁ REQUEJO, A., PAZ FERREIRO, J. and TARQUIS, A.M. (2008b). Assessing soil surface roughness decay during simulated rainfall by multifractal analysis. Nonlinear Processes in Geophysics, 15 (3): 457-468.

VIDAL-VÁZQUEZ, E., PAZ-FERREIRO, J., VIEIRA, S., TOPP, G., MIRANDA, J. and PAZ GONZÁLEZ, A. (2012). Fractal description of the spatial and temporal variability of soil water content across an agricultural field. Soil Science, 177 (2): 131-138

VIDAL VÁZQUEZ, E, CAMARGO, O. A., VIEIRA, S. R., MIRANDA, J.G.V., MENK, J.R.F., SIQUEIRA, G.M., MIRÁS AVALOS, J. M. and PAZ GONZÁLEZ, A. (2013). Multifractal analysis of soil properties along two perpendicular transects. Vadose Zone Journal, 12: 1-13. doi: 10.2136/vzj2012. 0188.

Van RAIJ, B., de ANDRADE, J. C., CANTARELLA, H. and QUAGGIO, J. A. (2001). Chemical analysis for evaluation of tropical soils fertility. Instituto Agronômico. Campinas. Brazil. 284 pp.

VIEIRA S.R., HATFIELD, J. L. NIELSEN, D. R. and BIGGAR. J. W. (1983). Geostatistical theory and application to variability of some agronomical properties. Hilgardia, 5:1-75.

VIEIRA, S.R., and PAZ GONZÁLEZ, A. (2003), Analysis of the spatial variability of crop yield and soil properties in small agricultural plots. Bragantia, 62: 127-138.

VIEIRA, S.R., de CARVALHO, J.R.P, and PAZ GONZÁLEZ, A. (2010). Jack knifing for semivariogram validation. Bragantia, 69: 97-105.

ZELEKE, T.B. and B.C. SI. (2006). Characterizing scale-dependent spatial relationships between soil properties using multifractal techniques. Geoderma, 134:440-452.